Quasicrystal

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http://homepage.mac.com/whitby/

http://web.me.com/whitby/Octahedron/Welcome.html

Reference

Octahedron1stEd.pdf–bookmark QUASICRYSTAL–pages 115-148

Introduction

This material is excerpted from *Octahedron*.

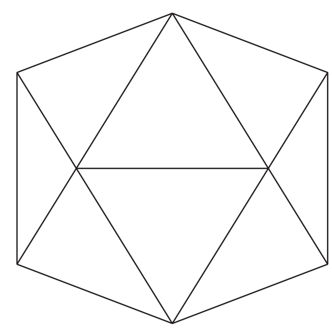
QUASICRYSTAL

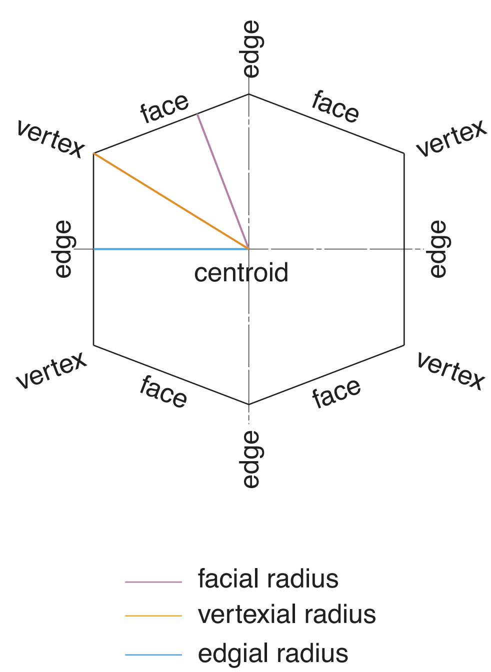
An epn of any atom of any group of any assembly in a crystal has an orientation which is identical with that of any other epn in that crystal. That is not true for the quasicrystal. To produce the quasicrystalline forms using identical regular octahedra requires that one octahedron be rotated with respect to another. The octahedra must still be joined by their polar edges, and they must be in structurally stable assemblies.

Fivefold forms

Icosahedron

A rigid regular icosahedron may be constructed using rigid equilateral triangular panels joined by hinges. Each of the eight faces of the regular octahedron is an equilateral triangle. Each face is defined by three edges. If the edges are polar, then one octahedron can be joined to another octahedron by a pairing of polarly attractive edges. The join is a hinge which permits rotation only about the shared edge. This pair can be added to so that a rigid icosahedron is produced in which each of the twenty panels is a regular octahedron which is polarly joined to three neighboring octahedra by the three edges defining one of its faces.





Geometry of the icosahedron.
The top figure shows an edgial view of the regular icosahedron. The bottom view shows the perimeter with topographic labels. Each of three radii are shown in color.

Geometry of the icosahedron

The icosahedron has twenty faces which are equilateral triangles. It has twelve vertexes and thirty edges. The icosahedron is viewed in the direction of an edgial diameter in the next figure.

For an edge length of s, the radial distances of the topographical features from the centroid of the icosahedron can be calculated.

In the figure, this radius projects as its true length for the two edges which are projected at their true lengths, one on the left and one on the right. The same is true for the two edges which project as points at the top and bottom of the figure. All these edges are labeled.The edgial radius is $r_{edge} = s\frac{1+\sqrt{5}}{4}$.

There are four vertexes which are labeled in the figure. Each lies on the same plane as the centroid and its radial distance is $r_{vertex} = s\sqrt{\frac{5+\sqrt{5}}{8}}$.

There are four faces whose planes are parallel to the viewing direction and which are viewed edge on in the figure. The shortest distance between a face and the centroid is perpendicular to the face and intersects the face at its centroid. This radial distance is $r_{face} = s\sqrt{\frac{7+3\sqrt{5}}{24}}$.

The three radial directions which are marked in the figure lie on the same plane. The angle between the vertexial radius and the facial radius is $r_{vertex} \wedge r_{face} = \operatorname{atan}\frac{1}{\sqrt{\frac{7+3\sqrt{5}}{8}}}$.

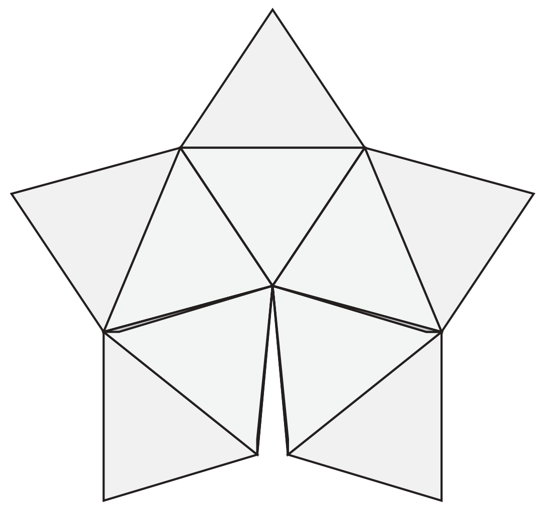
The vertexial radius and the edgial radius make an angle which is equal to $r_{vertex} \wedge r_{edge} = \operatorname{atan}\frac{2}{1+\sqrt{5}}$.

The facial radius and the edgial radius make an angle which is

$$r_{face} \wedge r_{edge} = \operatorname{atan}\frac{1}{\sqrt{\frac{7+3\sqrt{5}}{2}}}$$

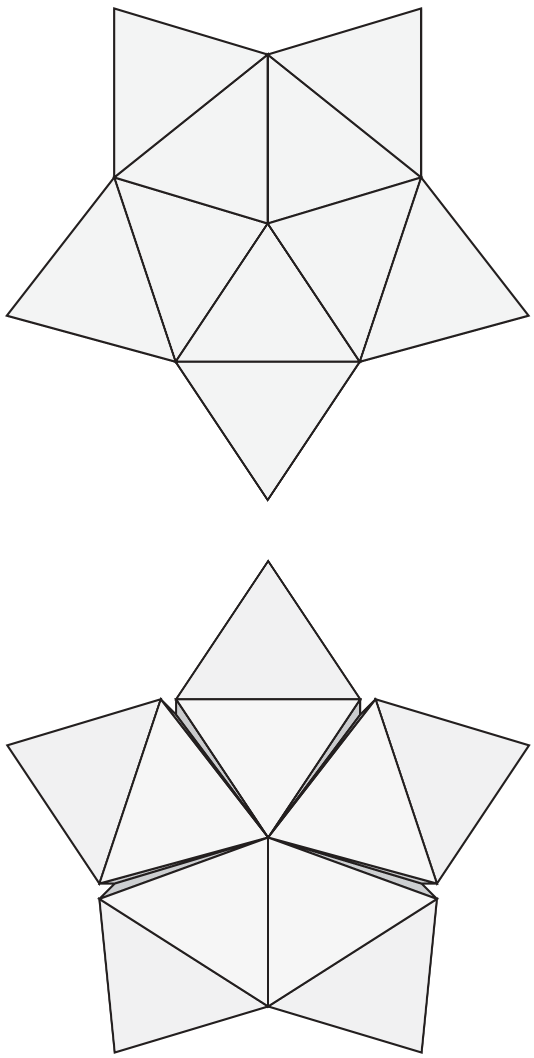
Icosahedral association of regular octahedra

For the regular octahedron, the angle between faces at a vertex is $\operatorname{atan}\sqrt{8}$. When five regular octahedra are joined face to face at a common vertex, the result is shown in the next figure. There remains a gap of $360° - 5\operatorname{atan}8$, which is approximated by $7°21'22''$.



Five octahedra joined facially.
The figure shows five regular octahedra joined facially so that each of the octahedra shares a vertex in common with the others/ It is seen that there remains a gap so that the ring is incomplete.

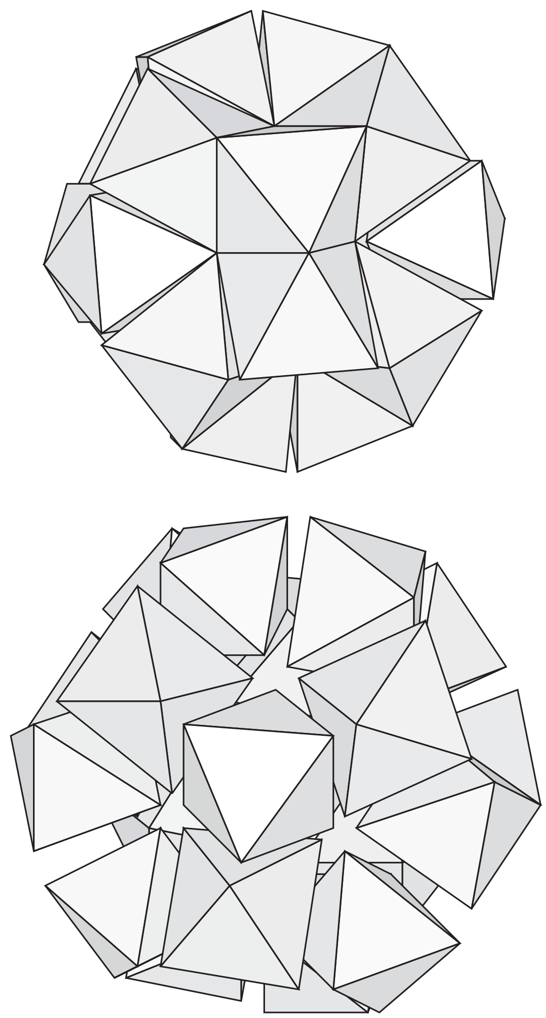
Twenty regular octahedra can form a regular octahedron when they join so that each contributes a face to a regular icosahedron. The relationship between the five octahedra at an icosahedral vertex is depicted in the next figure.



Two views of five octahedra from the icosahedral assembly below.
The upper view is of the inside and shows the join edges; the upper is of the outside and shows the separation of the outer edges.

The top portion of the figure is a view from the inside of the icosahedron in the direction of a vertexial radius. It shows that the edges of the octahedra are in contact, so that there is a join between adjacent octahedra. The lower view of the same five octahedra as seen from the outside in a direction opposite to the same vertexial radius shows the separation of the outer edges.

The full assembly of the icosahedral assembly of regular octahedra is shown in the next figure. In the upper view, the octahedra are in contact and appear as they would in the assembly. The lower view shows the twenty octahedra in the same orientation, but they are displaced radially to allow them to be more easily differentiated.

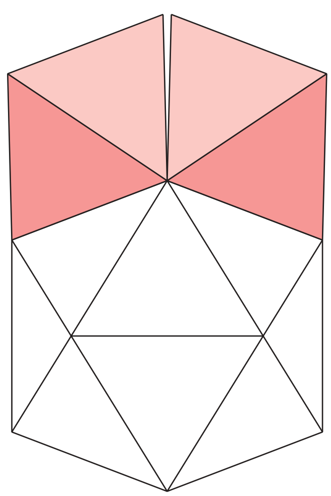


Twenty octahedra forming a regular icosahedron.
The lower view is an exploded view. Both are viewed in the direction of a face of the inner icosahedron.

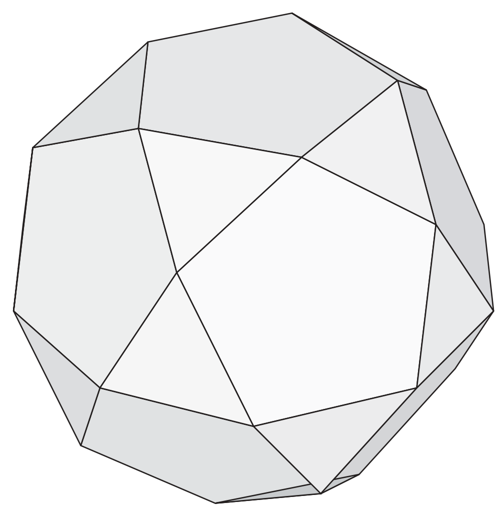
Each of the octahedra of the icosahedral assembly is a panel. Each of the edgial joins between a pair of panels is a hinge, but the hinges are stabilized by the other panels so that the resulting assembly is a rigid structure. In the figure, an edgial view of the icosahedron is shown with two of the octahedra at an edge. Each edge of the icosahedron is a join between two octahedra. The view shows the gap between the faces of the adjoining octahedra.

Icosidodecahedron

The face that each octahedron contributes to the icosahedron is turned towards the center of the assembly. It is paired with a parallel outward face of the same octahedron. Each of these outward faces defines a facial plane of a regular icosahedron. Each of the edges of each of the outer octahedral faces helps to define one of twelve facial planes of a regular pentagonal dodecahedron. Each dodecahedral face is defined by an edge of five different octahedra..The polyhedron which includes the twenty triangular faces of the icosahedron and the twelve pentagonal faces of the dodecahedron is called an icosidodecahedron.



Icosahedral assembly: gap between octahedra at edge.



Icosidodecahedron

Octahedral atoms form icosahedral assemblies

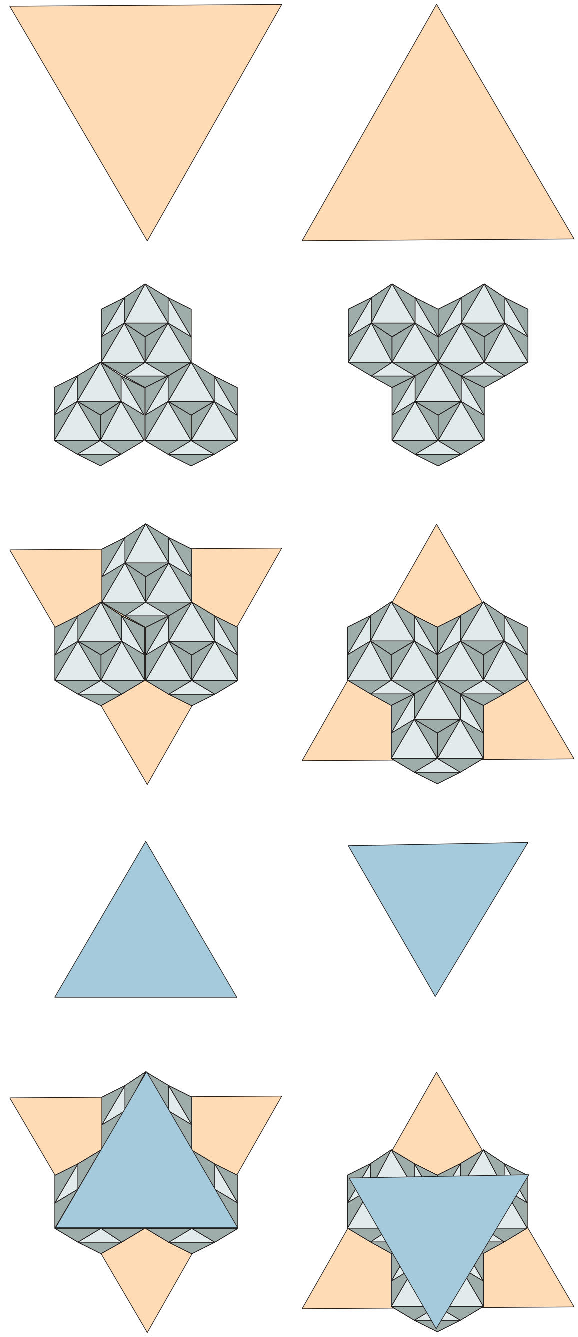
Each of the octahedra in the depicted icosahedral assemblies can be replaced by an atom or group of atoms of whatever complexity. Its volume must be contained within an octahedral volume. It must have He-octa edges which are colinear with the three edges of a face of the octahedral volume. These edges can join with identical edges of identical groups to form an icosahedral assembly.

Buckminster fullerenes

A number of Icosahedral groupings have been discovered which are composed of C-atoms. These are called *Buckminster Fullerenes*. A description of some of the assemblies follows.

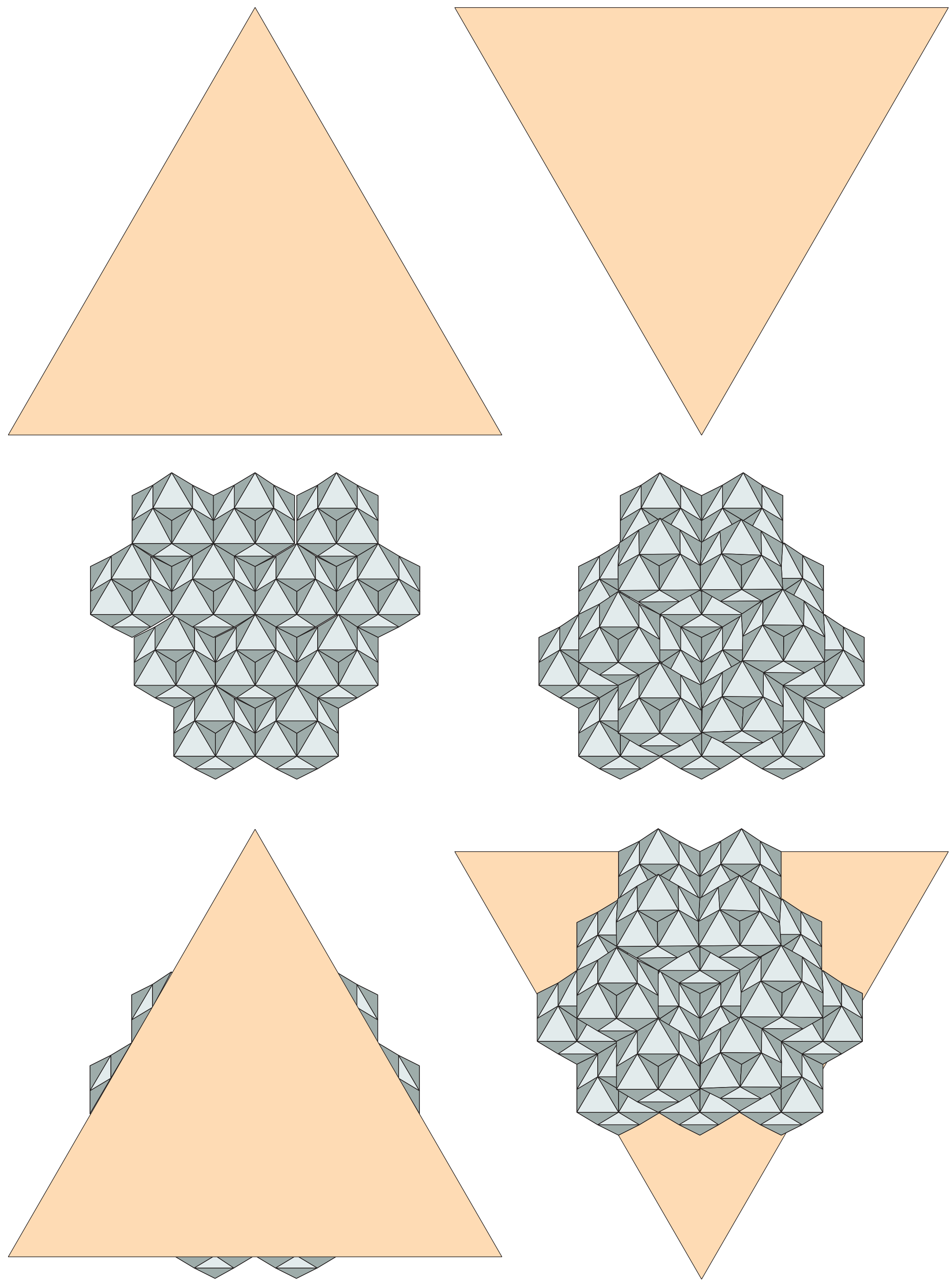
An icosahedral assembly of C-atoms

A C-atom is shown in the next figure with the two triangular faces it can form if it is the panel of an icosahedral assembly. The C-atom acts as the larger panel when its central tetrahedral void faces outward from the icosahedral centroid. The left column of figures shows the icosahedral face at the top. The next drawing is the C-atom as it would appear looking towards the icosahedral centroid. The third drawing shows the C-atom atop the icosahedral face. The fourth drawing is the icosahedral face that would result from using the upper face of the C-atom as a panel. In the bottom drawing the two faces sandwich the C-atom. The second column shows the same large icosahedral face at its top. The C-atom is inverted from the C-atom depicted on its left. The next drawing of the right column shows the inverted C-atom atop the large triangular face.The edges which are used to join the C-atom to other C-atoms to create this large triangle are colinear with the edges of the large triangle in this drawing. The small triangle is next in this column. In the last drawing of the column the C-atom is sandwiched between the two triangles. The edges of the small triangle on the left sandwich are colinear with the edges of the C-atom which are used to join to other C-atoms to form an icosahedral assembly. The two triangles have edge lengths which are in the ratio of 3 He-octa edges to 2 He-octa edges.



Icosahedral faces defined by C-atom.

An icosahedral assembly of C_6-rings



Icosahedral face defined by C_6 ring.

An icosahedral assembly of C_6-rings is formed in a similar manner. In the next drawing the C_6-ring is shown under the panel in the left column, and over the panel in the right column. The view at the bottom of the left column is how the panel would appear looking radially out from the centroid of the icosahedron. The drawing at the bottom of the right column is how the panel would appear looking in toward the centroid. of the icosahedral assembly. The size of the triangular face is six He-octa edges.

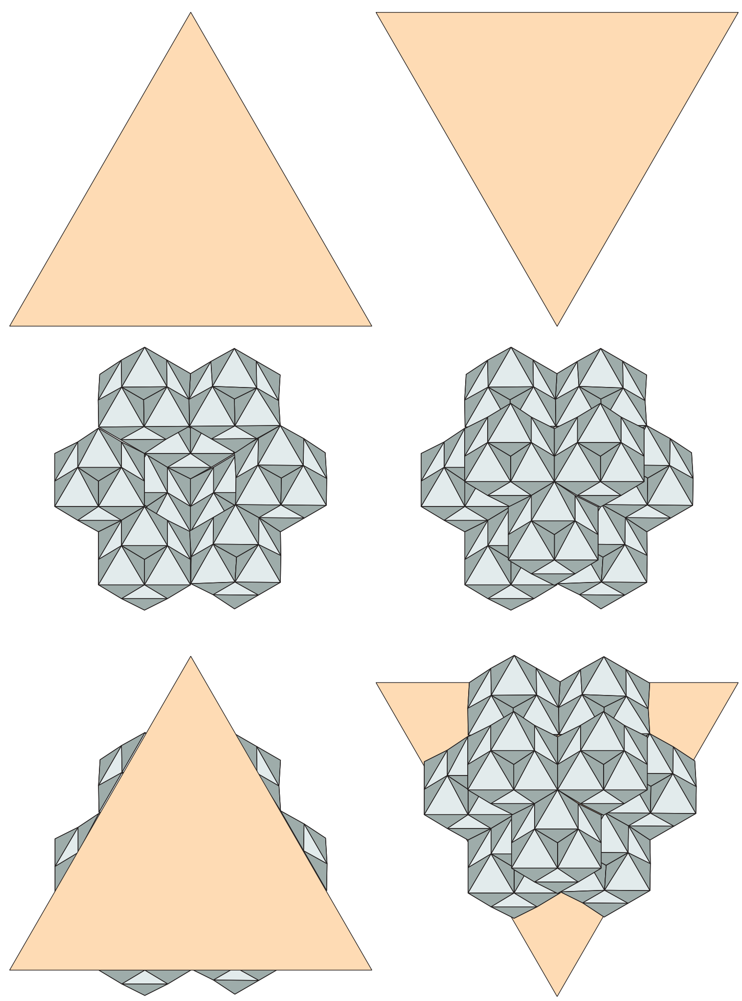
An icosahedral assembly of C_3-tetrahedra

Another C-atom group can form an icosahedral assembly. The group is composed of three C-atoms. Each C-atom supplies a face of a regular tetrahedron. The fourth face is open. A He-octa of each of two C-atoms combine to form the hinged edge of the icosahedral panel. The resulting icosahedral panel has an edge length of four He-octa edges.

An icosahedral assembly of C_4-tetrahedra

The C_4-tetrahedron has a C-atom panel for each face. This unit can make an icosahedral assembly. The join between icosahedral panels is provided by a He-octa edge of one of the C-atom panels of the C_4 tetrahedron. This is one of the ways in which a C-atom can make a panel of an icosahedron and has been depicted previously.

Icosahedral face defined by C_3 tetrahedron



Icosahedral assemblies with identical facial panels

A consideration of icosahedral assemblies which are composed solely of C-atoms suggests the listing in the next table. The number of C-atoms in each assembly is an integral multiple of twenty. The one with the fewest atoms contains just one atom per facial panel[1]. The most common assembly contains sixty atoms.

Table 14: Icosahedral assemblies

Atoms	Facial panel	Description
20	C-atom	
40	C_2-octa	Mg-octa
60	C_3-tetrahedron	Graphite CFU
80	C_4-tetrahedron	Diamond CFU
120	C_6-ring	Benzene ring

Icosahedral assemblies with more than one type of facial panel

Seventy atom assembly

A seventy atom icosahedron might be assembled from ten C_4-panels and ten C_3-panels. They would supply forty plus thirty C-atoms to give the seventy C-atoms required. Although these panels might be distributed in an irregular manner, they could be distributed regularly. One set could be in two five-panel groups at opposite vertexes with the other set supplying the ten equatorial panels.

Seventy-six atom assembly

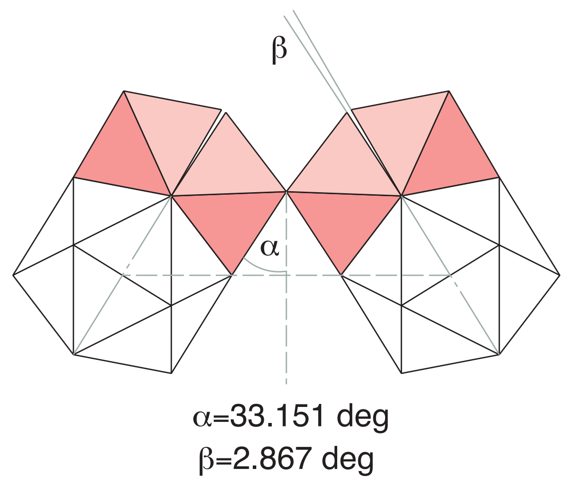
Seventy-six atoms can be obtained from sixteen C_4-panels and four C_3-panels. The sixty-four atoms of the C_4-panels plus the twelve atoms of the C_3-panels gives seventy-six C-atoms. These could be symmetrically distributed if the four C_3-panels were paired at diametrically opposed icosahedral edges.

1. A *New York Times* article describes the twenty C-atom assembly as "the smallest of the buckyball family". See "A Prodigious group and Its Growing Pains" by Kenneth Chang in the Science Times section of the *New York Times* of 10 October 2000.

Seventy-eight atom assembly

Two C_3-panels could occupy a pair of opposed icosahedral faces for six C-atoms. Eighteen C_4-panels could complete the seventy-eight atom icosahedral assembly.

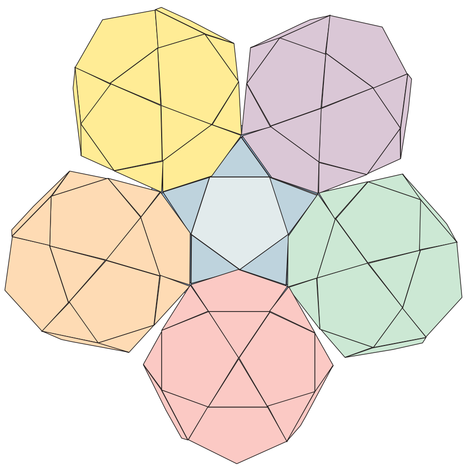
Icosidodecahedral joins



Icosahedral geometry: octal edgial contact between icosahedral assemblies.

Each dodecahedral plane is defined by an identical regular pentagonal array of five octahedral edges. A pair of identical icosahedral assemblies of regular octahedra may join so that a pentagonal array of octahedral edges of one assembly is congruent with an array of the other assembly to produce a rigid polarly attractive join. Again, each edge-to-edge join is stabilized by the other edge-to-edge joins. The geometry of the edge-to-edge join between a pair of octahedra belonging to adjoining icosahedral assemblies is shown in the figure. The two icosahedral centroids are joined by a line which is colinear with a vertexial radius of each assembly.

Because each of the pentagonal edgial arrays is rotated 36° to the array which is diametrically opposite, the move from one position to another requires a rotation of 36° between the joining units so that the octahedral edges at the join congrue. In the next figure, this rotation is seen in each of the five icosidodecachedra which have been joined to the central icosidodecahedron.

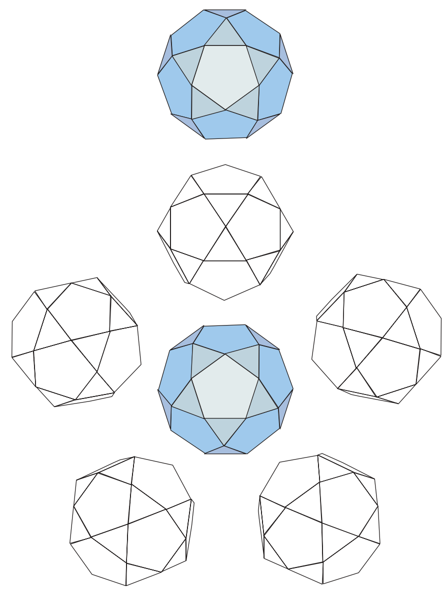


Icosidodecahedral hub showing five units joined to a central unit.

Orientations of the icosidodecahedron

When icosidodecahedra join pentagonal face to pentagonal face, there are twelve possible sites on a given reference unit. The joining unit is rotated 36° relative to the base unit about the axis joining their centroids. This axis is perpendicular to the facial join. It follows that the units joined in a given join direction alternate between two orientations. It is seen also that the pentagonal faces are arrayed as six pairs whose face normals are colinear and pass from facial centroid to facial centroid through the icosidodecal centroid. Each unit of each pair has the same orientation. This orientation differs from each of the other pairs. Thus, there are six possible orientations for the units joined to the reference unit. There is the orientation of the reference unit itself. There are, then, no more than seven possible orientations in an assembly of icosidodecahedra joined pentagonal face to pentagonal face.

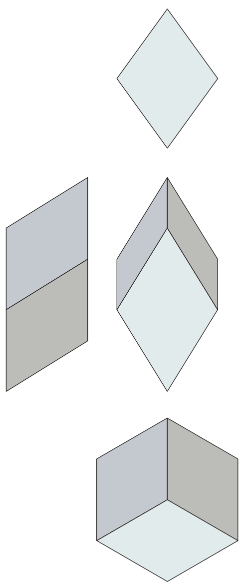
In the figure, an icosidodecahedron viewed normal to a pentagonal face is surrounded by icosahedra in the six other orientations. Together, these are the seven possible orientations.



Orientations of the icosidodecahedron.

The acute Penrose rhombohedron

Each icosahedral face connects three icosahedral vertexes. The three move directions through these vertexes provide the directions for the edges of a rhombohedron each of whose rhombic faces has an acute angle which is exactly the $\operatorname{atan} 2$. This shape is known as an *acute Penrose rhombohedron*. This shape can act as the unit cell for the five-fold figures that result from the joining of icosidodecahedra. The figure here shows the principal views of the rhombohedron. The top right view shows the angle between the faces at an edge to be $72°$ or its supplement.



Acute Penrose rhombohedron: principal views.

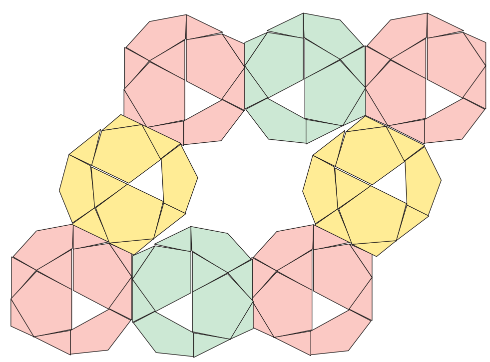
The view on the left is along an edgial diameter. The top view on the right is parallel to the planes of four of the faces and shows the angle between the faces at an edge to be $72°$ or its supplement. The middle figure is normal to a face and shows its true shape. The bottom figure

The edges of the rhombohedron have the same length. They are of two types. One type ends at the three-fold axis of symmetry; the other type does not. The former are in two sets of three. The others join the ends of the opposed axial sets and are six in number. These comprise the twelve edges of the rhombohedron.

The vertexes are of two types. Two are on the axis of three-fold symmetry. Six others are in sets of three at one third and two thirds of the distance between the vertexes on the threefold axis. The view along the three-fold axis is at the bottom of the righthand column.

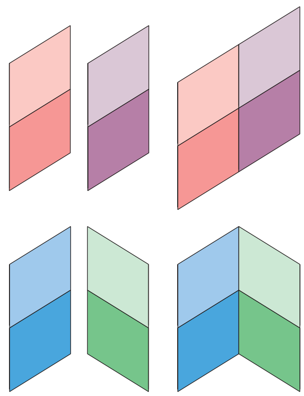
Rhombohedral assembly of icosidodecahedra

To realize the acute Penrose rhombohedral shape with the icosahedral assemblies, it is necessary to have assemblies of the same orientation at each of the rhombohedral vertexes. That requires mid-edge units to connect them to restore the orientation. (Twice 36° is 72° which for a pentagonal array means congruency.) In the figure the units are tinted according to their orientations. The red tinted are each at a vertex. The mid- edge units are differently oriented.



Face of an acute penrose rhombohedrdon defined by icosidodecahedra.

Rhombohedral joins



Rhombohedral joins.
Rhombohedra join face to face in one of two ways. The rhombohedra here are viewed edgially. In the top row, the two rhombohedra on the left have the same orientation. They are shown joined on the right. This is the simple crystalline join. The two rhombohedra on the left of the bottom row have different orientations. The resulting join is shown to the right. This is a twinned crystalline join.

The five-fold forms are modelable using this form with these two joins.

Crystalline join

A rhombohedron can be joined to an identical rhombohedron in identical orientation so that a face of one is congruent with a face of the other. This join is shown in the top row of the figure. This is a typical join for an untwinned crystal.

Twin join

A rhombohedron can be joined to an identical rhombohedron so that the face of one is congruent with the face of another while the two have different orientations. This is a twinned structure. It is shown in the bottom row of the figure.

Icosahedral assembly of acute Penrose rhombohedra.

Twenty acute Penrose rhombohedra can be joined face to face so that they share a common vertex. For each rhombohedron, the common vertex on its three-fold axis. The axis is normal to a face of a regular icosahedron.

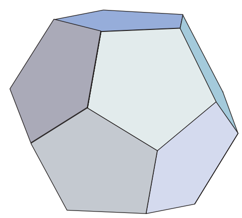
In the figure, each of the yellow rhombi is the projected face of a rhombohedron viewed parallel to four of its faces. Each of the five pairs of blue rhombi belong to a rhombohedron which is face joined to the yellow rhombohedron above it. Each of the green rhombi are face joined to each of the two adjacent blue rhombohedra.

The outermost vertexes of the twenty rhombohedra are at the vertexes of a regular pentagonal dodecahedron. They define the twelve faces of the dodecahedron. The five yellow rhombohedra share an edge which is an axis of fivefold symmetry. This axis is perpendicular to a dodecahedral face at the facial centroid.

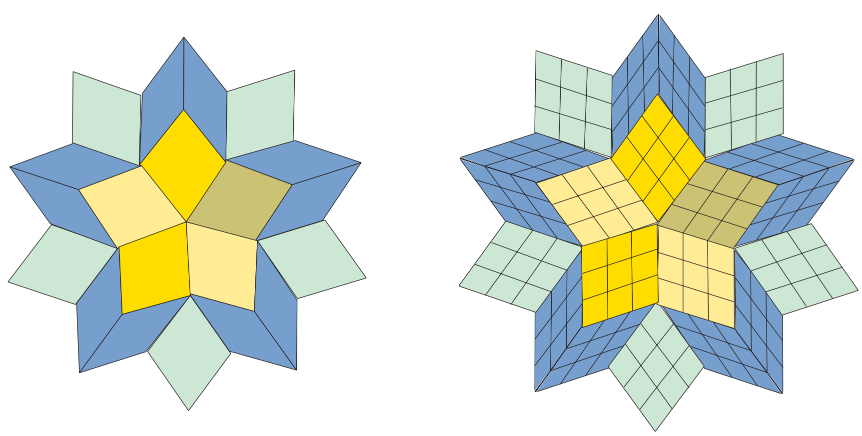
Crystalline expansion of five-fold forms with acute Penrose rhombohedral unit cells

Crystalline expansion of the rhombohedron

The crystalline expansion of the twenty rhombohedra maintains the geometry of the assembly while revealing the twinning. The figure shows twenty compound rhombohedra in the same relationship as the simple rhombohedra shown previously. Each of the compound rhombohedra is composed of twenty-seven simple rhombohedra.



Pentagonal dodecahedron



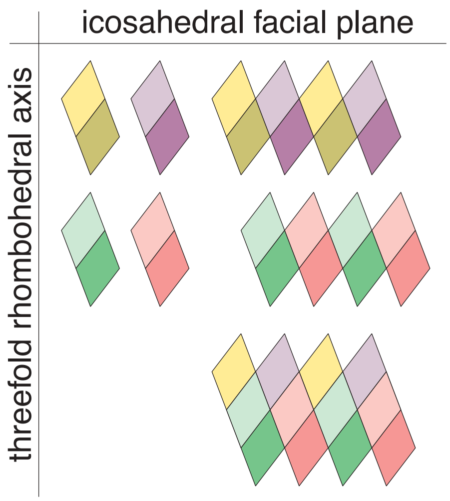
Twenty acute penrose rhombohedra joined facially with a common vertex.
The figure shows two assemblies of acute penrose rhombohedra. The one on the left is composed of simple rhombohedra; the one on the right is composed of compound rhombohedra. In each case, there are twenty rhombohedra which share a common vertex.

Crystalline expansion of the icosahedron.

The icosahedral plane is defined by vertexes on the threefold axis of the rhombohedra. The threefold axes of each rhombohedron is perpendicular to the icosahedral plane. A rhombohedron in a layer parallel to the icosahedral plane shares an edge with as many as six adjacent rhombohedra. It is facially joined to as many as three rhombohedra in an adjacent layer. The rhombohedra in the icosahedral plane are in the same orientation.

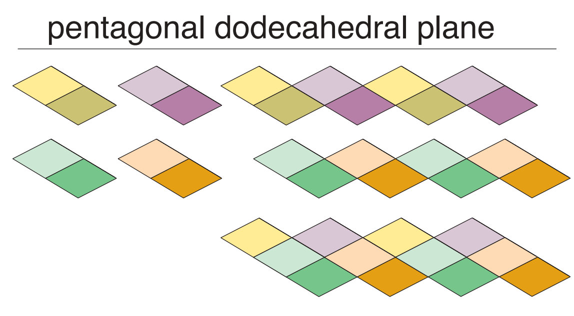
Crystalline expansion of the pentagonal dodecahedron

The pentagonal dodecahedral plane is defined by vertexes of the rhombohedra which terminate the minor diameters of the rhombohedral faces. The rhombohedra are in layers which are parallel to the dodecahedral plane. The layer is parallel to the major diameter of each of four faces of the rhombohedron. Each rhombohedron in the layer shares an edge with as many as six adjacent rhombohedra in the layer. Each rhombohedron is facially joined to as many as three rhombohedra in each adjacent layer.



Icosahedral plane.

This is the arrangement of the acute Penrose rhombohedra at an icosahedral surface. The rhombohedra are viewed parallel to a pair of opposed faces. The threefold axis of each is perpendicular to the icosahedral surface. In the two upper rows, the units on the left are individual rhombohedra. They are arranged on the right in edge to edge contact. The bottom row shows them joined facially.

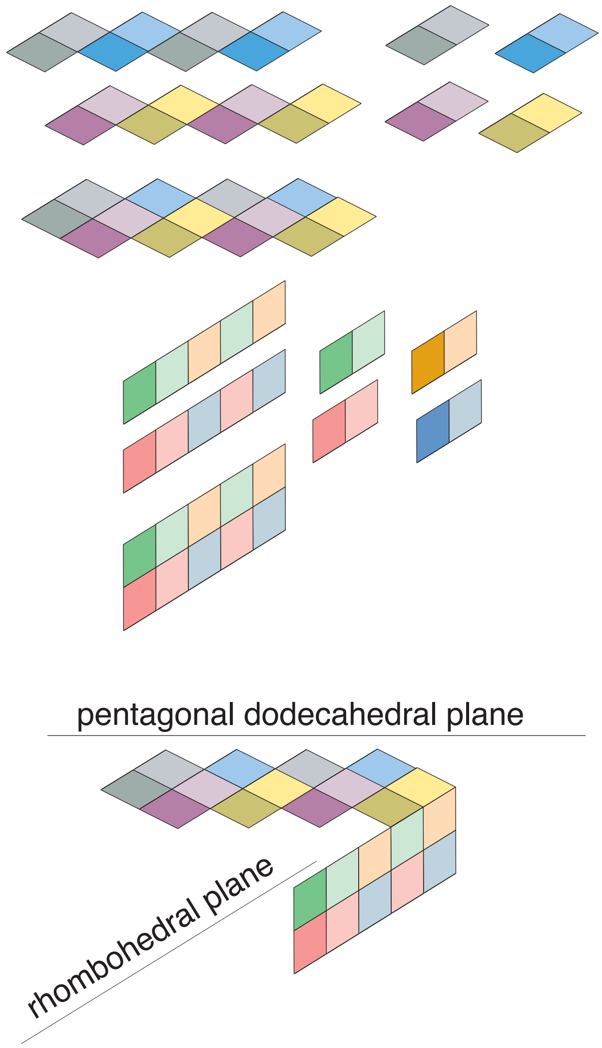


Pentagonal dodecahedral plane.

This is the arrangement of the acute Penrose rhombohedra at a pentagonal dodecahedral surface. Each rhombohedron is viewed parallel to a pair of opposed faces. In the top two rows, the two units on the left are individual rhombohedra. Their arrangement in the assembly is shown on the right, where the rhombohedra are in edge to edge contact in two lines parallel to the pentagonal plane. The lines are shown in facial contact in the bottom row.

Join between rhombohedron and dodecahedron

In the next figure, the arrangement of the Penrose rhombohedra are shown for each of two crystalline regions. The upper group shows the arrangement within the region of the pentagonal dodecahedral plane. The middle group shows the arrangement in the region of the rhombohedral plane. The threefold axes of these rhombohedra are parallel to a vertexial radius of an icosahedron. In each of the two regions, the rhombohedra are in the same orientation and the facial joins are crystalline. At the junction of the two regions the joins are twins. This is seen in the bottom grouping.

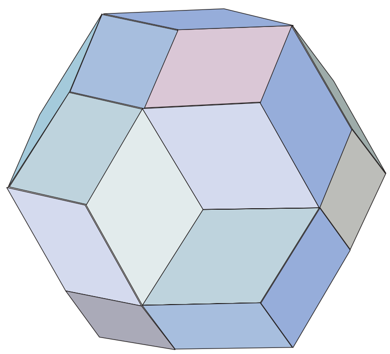


Junction of dodecahedral and rhombohedral planes.

The arrangement of the acute Penrose rhombohedra at the pentagonal dodecahedral plane is shown at the top. Next, their arrangement at the rhombohedral plane is shown. The arrangement of the rhombohedra at the junction is shown at the bottom.

The join the of the rhombohedra at the junction of the two planar regions is of the twin type. Within each region, the join is crystalline.

The triacontahedron



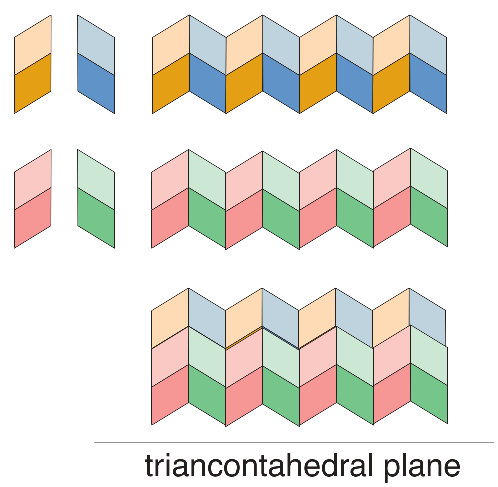
Triacontahedron.
The triacontahedron has thirty faces which are identical rhombi, one for each of the edges of the icosahedron. It has twelve five-fold vertexes, one for each vertex of the icosahedron. It has twenty threefold vertexes, one for each face of the icosahedron.

The triacontahedron is a fivefold form related to the icosahedron.

- •Each of the thirty identical rhombuses which constitute its faces is normal to a radius which is colinear with an edgial radius of a concentric icosahedron.
- •A facial radius of the icosahedron is colinear with the radius which intersects each face of the triacontahedron at its centroid.
- •The major diameter of each triacontahedral face is terminated at each end by a fivefold vertex which lies on the same line as a vertexial radius of the icosahedron.
- • Each of the major diameters of the triacontahedral faces can be colinear with an edge of a concentric icosahedron.
- •The minor diameter of each face is terminated by a threefold vertex which lies on the same line as a facial radius of the icosahedron.

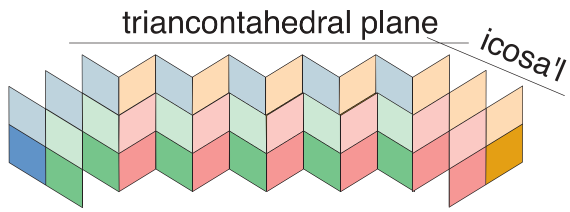
Crystalline expansion of the triacontahedron.

The triacontahedron can be formed by an assembly of acute Penrose rhombohedra. The triacontahedral plane is defined by threefold vertexes of the Penrose rhombohedra. These rhombohedra are facially joined in crystalline layers which are perpendicular to the triacontahedral plane and run parallel to the major diameter. They are facially joined as twins to rhombohedra in adjacent layers. In the figure, the arrangement of the rhombohedra within the layer which is parallel to the triacontahedral plane is shown. The development of the two layers are depicted in the upper rows. The bottom row shows the arrangement of adjacent layers. The twin joins within the layers parallel to the triacontahedral plane are evident in the upper rows; the crystalline joins in the layers perpendicular to the triacontahedral plane are evident in the bottom row.



Triacontahedral plane.
The acute Penrose rhombohedra which form the triacontahedral plane are oriented so that their facial join planes are perpendicular to the triacontahedral plane. Each of the upper rows has a pair of individual rhombohedra in the orientations which occur in the assemblies to the right. The combined assembles are at the bottom. The join rhombohedra in the row is that of twins; the join between rhombohedra of adjacent rows is crystalline.

Triacontahedral and icosahedral planes



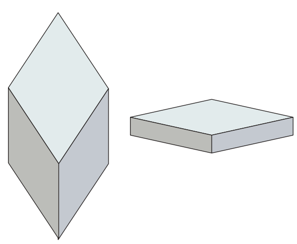
Relationship between the triacontahedral plane and adjoining icosahedral planes.
The view is normal to the minor diameter, parallel to the major diameter of the triacontahedral face.

The region of the triacontahedral plane is flanked by a pair of icosahedral planar regions. The view is parallel to the major diameter of the triacontahedral face. The two orientations of the rhombohedra of the triacontahedral region are seen in the icosahedral regions, within each of which the rhombohedra are identically oriented.

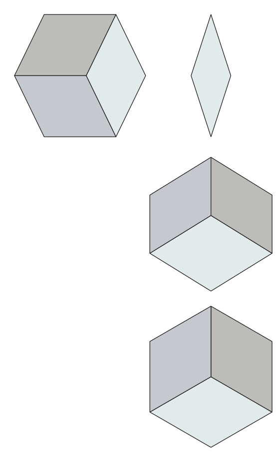
The obtuse Penrose rhombohedron

The obtuse Penrose rhombohedron has faces which are geometrically similar to the acute Penrose rhombohedron.

Where the faces of the two rhombohedra are identical, the length of the three fold axis of the acute rhombohedron is over four times that of the obtuse rhombohedron.



The Penrose rhombohedra
The two Penrose rhombohedra viewed perpendicularly to their threefold axes.



Obtuse Penrose rhombohedron.
In the right column: View parallel to four faces, top; view of true face, middle; view parallel to threefold axis, bottom.

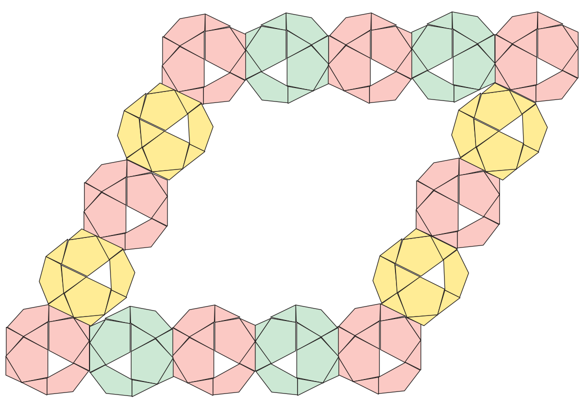
The view on the left is normal to the view to its right. The edges parallel to the top of the page are projected in their true length.

Face of obtuse Penrose rhombohedral assembly of icosidodecahedra

When icosidodecahedra assemble to form the edges of an obtuse Penrose rhombohedron, the resulting edge length is twice the length of the edge of the acute Penrose rhombohedron. This is so that the icosidodecahedra which define one edge of the rhombohedron does not obstruct the formation of another edge. Extending the edge of a rhombohedron composed of icosidodecahedra requires the addition of two units. This assures that the vertexial units are in the same orientation.

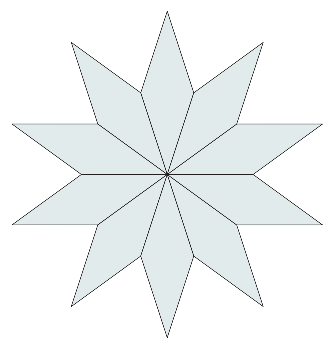
Tenfold star of obtuse rhombohedra

In the obtuse rhombohedron, the angle that each pair of faces makes at a mid-edge is 36°. When the obtuse rhombohedron is the unit cell for a quasicrystalline form, ten of the unit cells can join face to face so that they share an edge. This gives the appearance of a tenfold star. Each of the joins between adjacent rhombohedra is of the twin type.



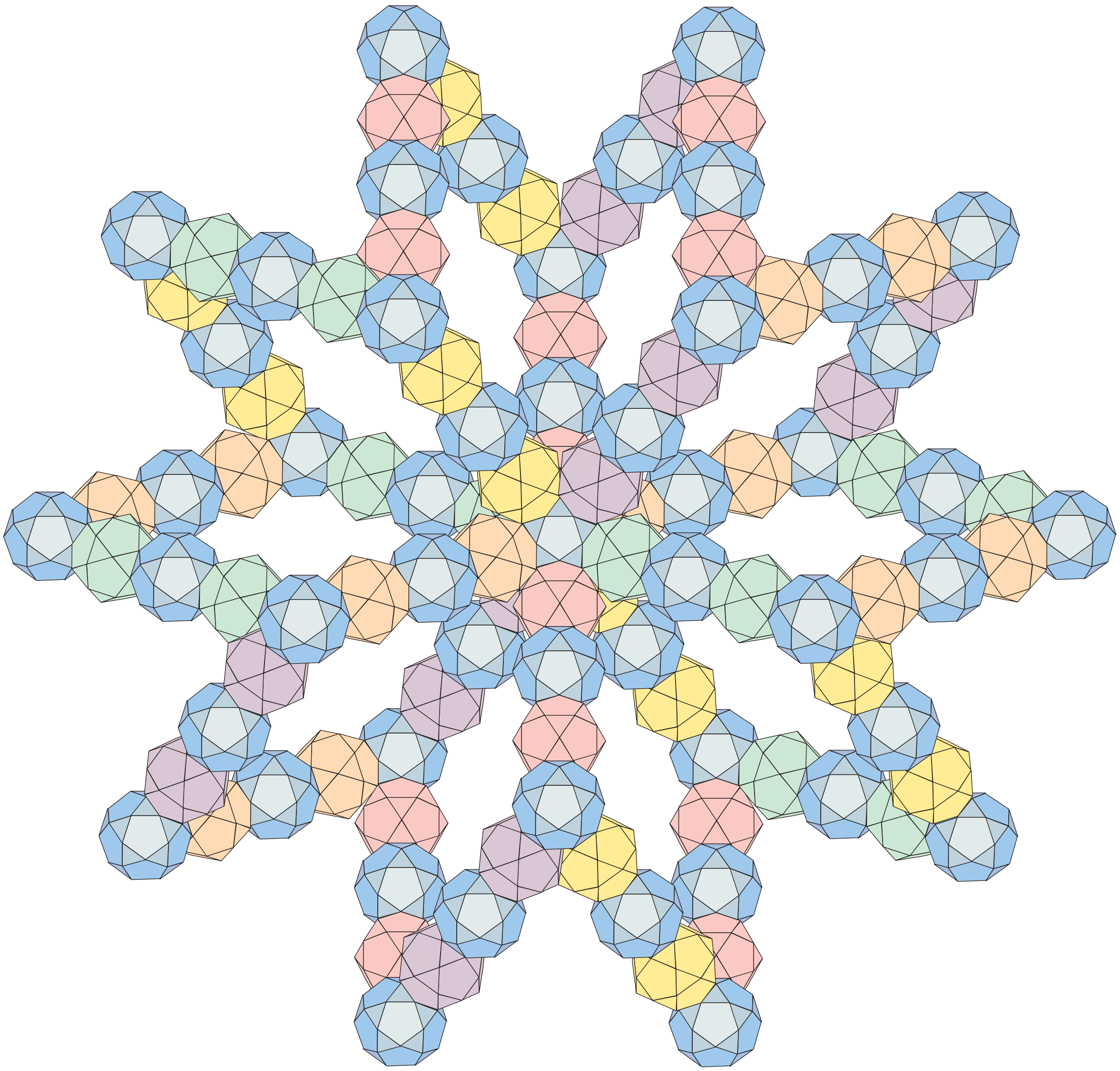
Face of obtuse Penrose rhombohedron defined by icosidodecahedra.

The icosidodecahedra with a common orientation have a common color.



Tenfold star of obtuse Penrose rhombohedra.

Tenfold star of icosidodecahedra

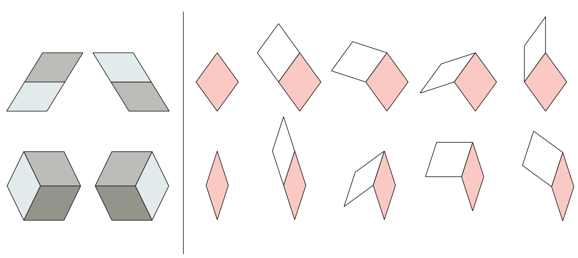


Decagonal assembly of icosidodecachedra.

The arrangement of icosidodecahedra in an assembly akin to the tenfold star is shown in the figure at the top of the next page. Each of the icosidodecahedra is colored according to its orientation. There is a central icosidodecahedron.

Joins between Penrose rhombohedra

There are two possible facial joins between a pair of Penrose rhombohedra. They differ by a rotation of 180 degrees about the normal to the join faces. In the figure, the red-tinted rhombuses are the projections of the reference rhombohedron and the white-filled rhombuses are the projections of the joining rhombohedra. These are the possible facial joins for the Penrose rhombohedra in the reference projections. The gray-shaded rhombohedra on the left are views of the reference rhombohedra normal to the viewing direction on the right. Both the acute rhombohedron and the obtuse rhombohedron can be in either of the two orientations shown and still project as the red-tinted rhombuses.



Facial joining of Penrose rhombohedra.
In the top row, the reference unit is the acute rhombohedron; in the bottom, it is the obtuse rhombohedron.

Icosahedral alloy $Mg_{32}(Al,Zn)_{49}$[1]

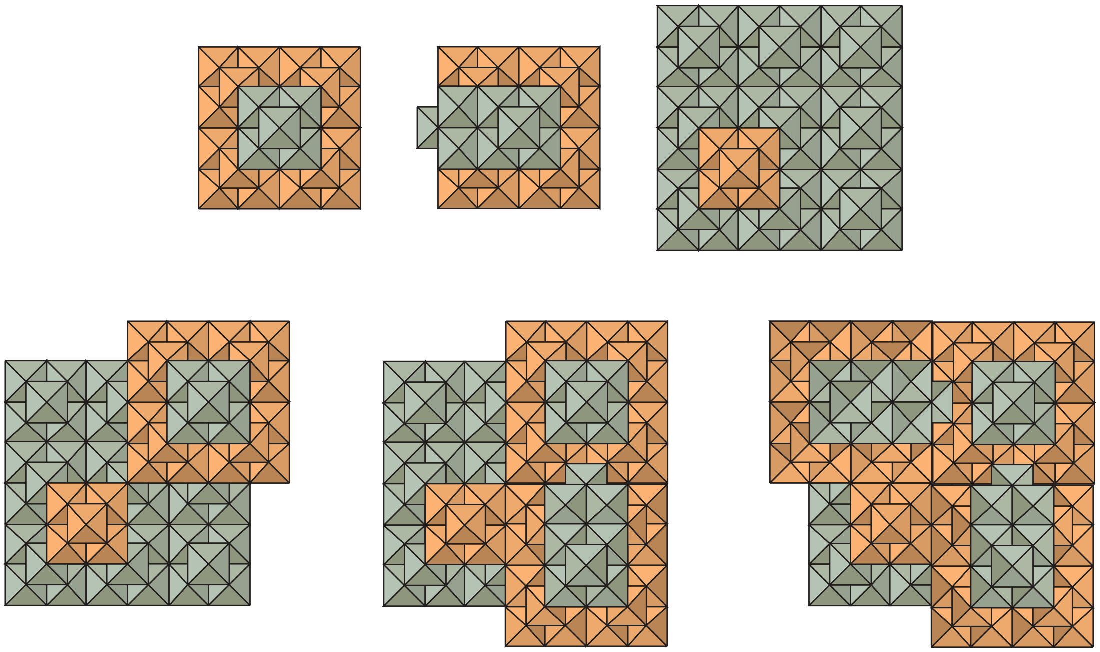
The alloy $Mg_{32}(Al,Zn)_{49}$ is of particular note in the study of quasicrystals. Efforts were made to assign the atoms to places within a Penrose rhombohedron because that form is useful for analyzing quasicrystals with fivefold symmetry. But that symmetry is due to the icosahedral assembly of regular octahedral subassemblies. The icosahedra assemblies define the Penrose rhombohedral unit cells. It is the icosahedral assembly, then, that must produce the atomic census of the alloy.

The seventy-one atoms of the alloy must be apportioned to the twenty octahedral assemblies which form the icosahedral assembly. Al-atoms and Mg-atoms occupy equivalent volumes. The volume is much less than that required for an Zn-atom. So, the Zn-atom is the likely base for the octahedral assembly. Using twenty Zn-atoms leaves twenty-nine Al-atoms and thirty-two Mg-atoms to be distributed among the twenty icosahedral assemblies. That requires that three of the smaller atoms be included with each of the Zn-atoms and leaves one atom to be allocated to one of the twenty icosahedral assemblies.

In the top row of the figure, the three types of atoms are shown in a view parallel to the vertexial growth axis. The Mg-atom is on the left, the Al-atom is in the middle, and the Zn-atom is on the right. In the bottom row, the left group is a Zn-atom to which a Mg-atom has been joined. The join is such that the He-octa of the Mg-atom fills the Fe-octa Se-octa void of the Zn-atom. Adding an Al-atom to the left group so that its He-octa occupies the Ge-octa void of the Zn-atom produces the middle group of the bottom row. To this group a second Al-atom is added so that its He-octa occupies the Kr-octa void of the Zn-atom.

The icosahedral face produced by these Zn-Mg-Al panels has an edge length of four He-octas. This is the same size face that is produced by the C_3-tetrahedron.

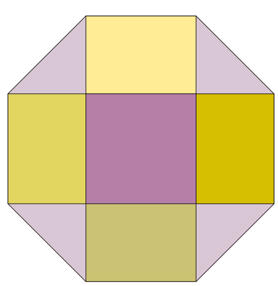
1. David R. Nelson, Quasicrystals, *Scientific American* August 1986 page 43.



Facial panel for $Mg_{32}(Al,Zn)_{49}$ icosahedral assembly.
The top row shows the three atoms which form the assembly.
The bottom row shows the formation of a $ZnAl_2Mg$ assembly which is a possible panel assembly.

Eightfold forms

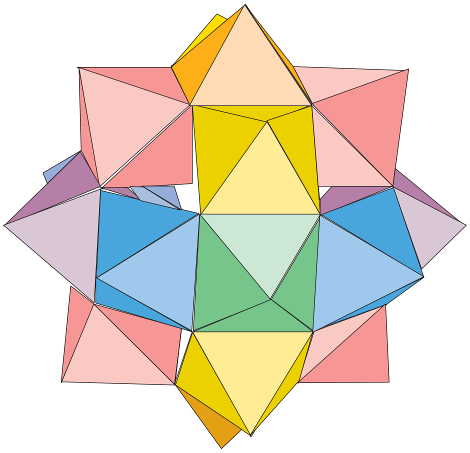
Rhombicuboctahedron

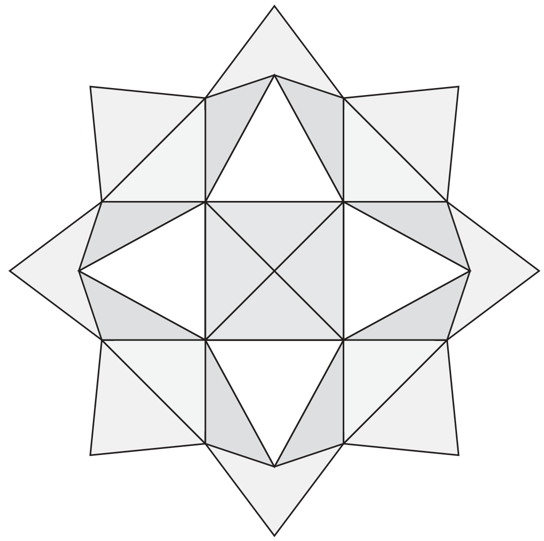


The rhombicuboctahedron has eighteen faces which are squares and eight faces which are equilateral triangles.It has twenty-four vertexes. The square faces are arrayed so that they may be seen as three mutually perpendicular rings. When each of the rings is viewed parallel to the planes of the square faces, their edges define a regular octagon. The square faces make an angle of 45° where they meet. This can be seen in the figure, where the perimeter is defined by the edges of the square faces which are normal to the plane of the projection. The view is normal to a square face which is common to two of the rings.

Rhombicuboctahedral assembly of regular octahedra

Eighteen regular octahedra joined edge to edge at an edgial equator will produce a stable rhombicuboctahedron. The edgial equator of each octahedron is a square and these account for the square faces of the rhombicuboctahedron. The eight triangular faces are defined by the unpaired equatorial edges of the octahedra.

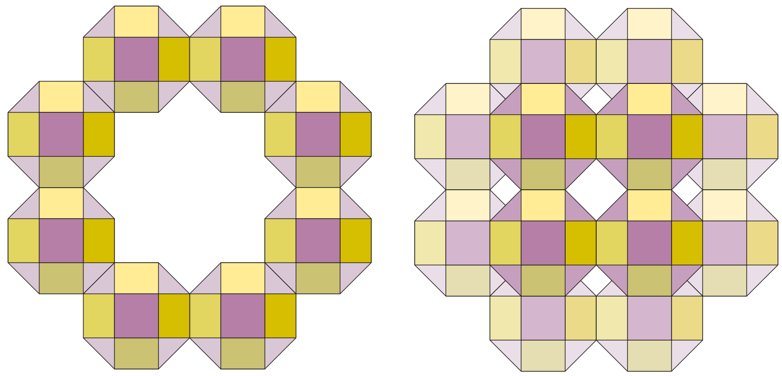




Rhombicuboctahedral assembly

The figures show two views of a rhombicuboctahedral assembly of eighteen regular octahedra. The view on the left is a perspective view, that on the right is an orthographic projection.

The square faces are arrayed as three mutually perpendicular and intersecting regular octagonal prisms. The octahedra which occupy the faces at the intersections of the octagonal arrays are edgially joined to each of four neighboring octahedra. There six of these positions. The other twelve share an edge with each of two intersectional octahedra. Each square face is matched with a diametrically opposite face. The octahedra in the two positions are in the same orientation. There are nine rotational orientations for the eighteen octahedra, so the octahedra are not in crystalline order. If each of the octahedra were replaced with a compound octahedron of six units each, and the outermost vertexial unit were removed from one octahedron and the innermost from its opposite, then none of the joins of the rhombicuboctahedron is disturbed and each square facial position is occupied by an identical assembly of five octahedra. This permits the rhombicuboctahedron to be joined face to face with an identical assembly so that the outermost vertexial subocta of one of the octahedra fits crystally into the position of the removed subocta on the opposed face of the other assembly and this produces an edgial join between the pair. The pair will be identically oriented.

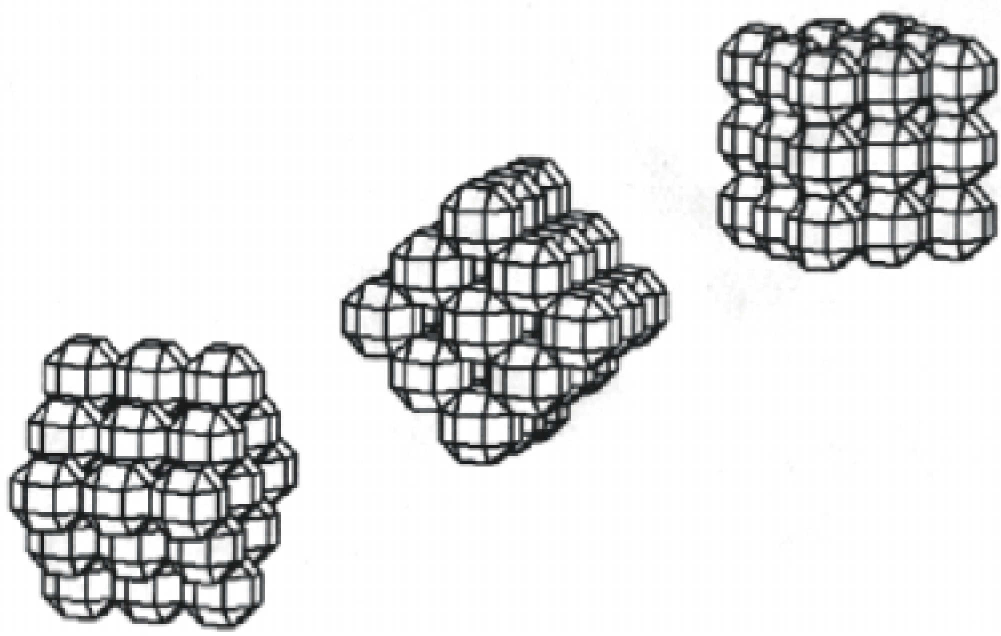


Ring of rhombicuboctahedra

Eight rhombicuboctahedra can assemble square face to square face to form an octagonal ring. Two of these rings can be stacked so that each unit of the first ring is joined to the unit of the ring above it by a square face. A ring of four rhombicuboctahedra can be made and place atop the joined rings and another below the joined rings. Together, these assembled rings form a larger compound rhombicuboctahedron. Each of the simple units occupies one of the twenty-four vertexes of the compound assembly. Twenty-four of these assemblies could be joined in similar fashion to produce a larger compound rhombicuboctahedron..

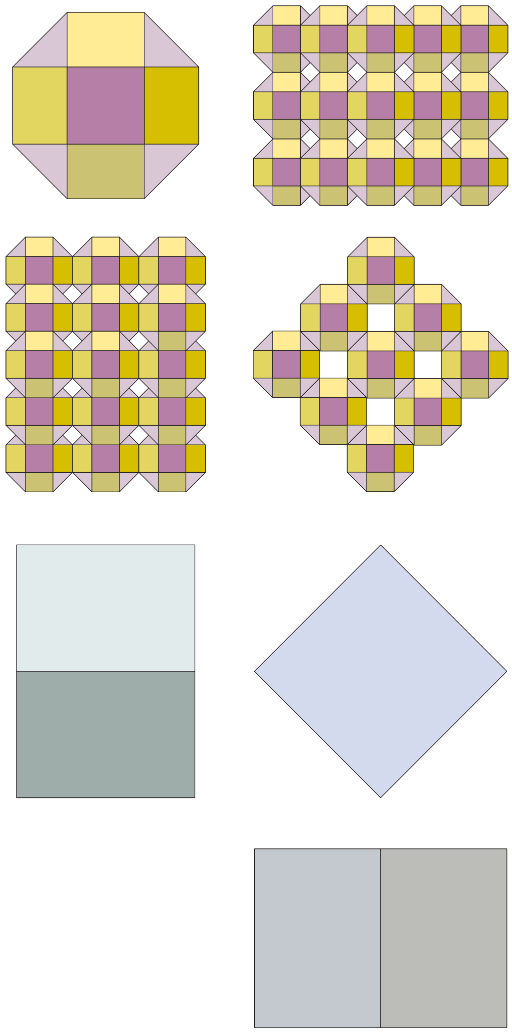
Cubes of rhombicuboctahedra

Each pair of opposed intersecting faces combine with four non-intersecting faces to define a cube. The effect of developing these cubes by joining rhombicuboctas only on the faces of one of the described cubes is seen in the next figure.



Three cubal assemblies of rhombicuboctahedra.

The figure shows three cubes of different orientation that are formed by rhombicuboctahedra each of which is identically oriented to each of those in each of the three cubes

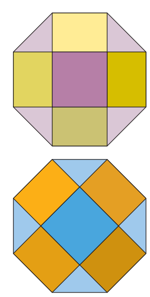


Rhombicuboctahedral cubic assemblies.
The rhombicuboctahedron depicted in the upper left is the unit which forms the three assemblies adjacent to it. Each is a cube whose faces are not parallel to the faces of the other two cubes. The shaded drawings at the bottom are projections of cubes which are in the same orientation as the assemblies above.

Each of the cubes is composed of identical subunits in identical orientation. But, the cubes produced are differently oriented. The three cubes are in the orientations required to produce the **quantum polyhedron**[1], in which the three cubes have a common centroid. Although each of the three cubes can be rotated so that its join directions are parallel to those of either of the others, to produce the quantum polyhedron requires the use of the three sets of join directions. The quantum polyhedron is polycrystalline; each of the cubes is crystalline.

Pseudo-rhombicuboctahedron

If an octagonal array of square faces of the rhombicuboctahedron is taken as a reference, then the other square faces are in groups of five on either side of the equatorial faces. Each of these groups is a mirror image of the other. One group can be rotated 45° to the other and joined to the equatorial group in this new position. This figure is stable and the square face positions can be occupied by the same octahedral groups as for the rhombicuboctahedron. To join the units in the directions of the rotated faces will require a rotation between the units.



Pseudo-rhombicuboctahedron.
The upper drawing is the top view of the pseudo-rhombicuboctahedron. The lower drawing is the bottom view. The bottom face and the four faces to which it is joined are rotated one-eighth turn to the top face and the four faces to which it is joined.

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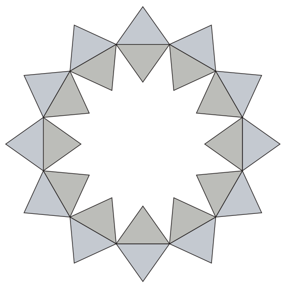
1. *Scientific American* Feb 93 The Artist, the Physicist and the Waterfall p. 30

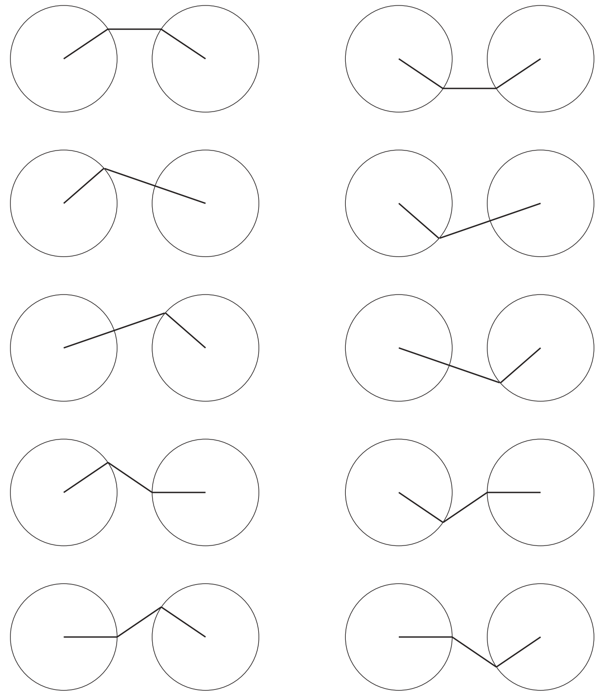
Twelvefold forms

A twelvefold assembly is possible if a regular dodecahedral ring can be formed using regular octahedra. Such a ring is depicted in the next figure. The two edges of each of the edg-



Dodecahedral ring of octahedra.

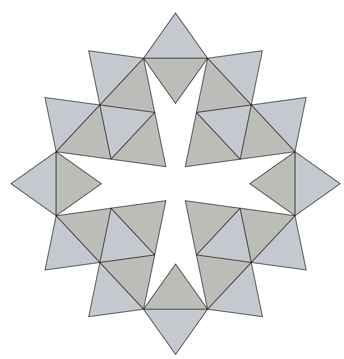
ially view octahedra which are shared by adjacent octahedra of the ring lie an edgial equator. The edges on this equator define a square. This ring must be angularly stablized by additional octahedral units. Two adjacent panels which are in turn adjacent to stabilized panels will themselves be stabilized. In the next figure, the instability of three adjacent unstabilized panels is seen in the next figure. The three panels are represented by line segments. The line segments within the circles are attached to stabilized panels and that hinged join is at the center of the circle. The other end of the panel must lie on the circle. The distance between the circles is fixed. The top left drawing shows the desired symmetry. The remaining drawings show the extremes of the dispositions that the three panels can assume within these constraints. The panels within the circles are limited to a rotation between the situations shown in the second and third rows respectively. In these positions the other two panels are colinear. The bottom two rows show the arrangement between the panels when one of the end panels is midway between its rotational limits.



Disposition of three rotatable edges.
The edges within the circles rotate about the centers of the circles. These centers are fixed relative to one another.

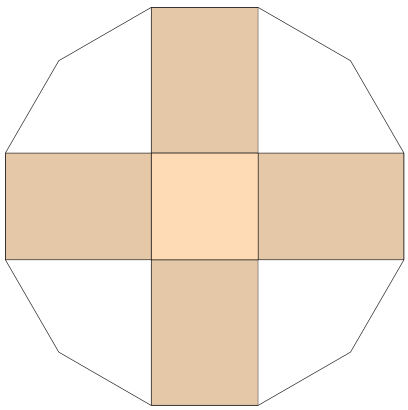
Stabilizing rings

The stabilizing ring assemblies for the dodecahedral ring are of two kinds. These are fourfold and threefold. The fourfold is a dual ring which has a square panel for its hub that is

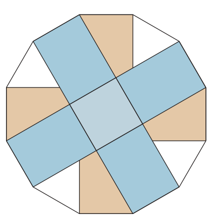


Stabilizer ring for dodecahedral ring.
The leftmost and rightmost octahedra are members the dodecahedral ring.

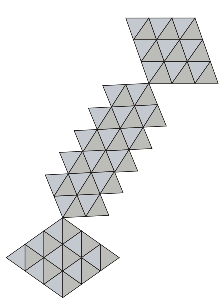
an edgial equator of an octahedron. Each edge of this equator is joined to one of the octahedra of the dodecahedral ring by a rectangular assembly of octahedra. In the next figure, one ring of the dual ring assembly is viewed parallel to its axis.The leftmost octahedron and the rightmost octahedron are on the dodecahedral ring. The octahedra at top and bottom lie on the axis of the dodecahedral ring. The next figure shows a view of the assembly normal to the dodecahedral ring which shows equators of its constituent assemblies.

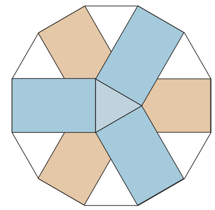


The fourfold stabilizer can be split in half and the halves rotated so that they stabilize different dodecahedral panels. This is shown in the next figure where each half has been colored to differentiate it.



The threefold stabilizing assembly has a face of a regular octahedron for a hub. Its legs are rectangular and join to the dodecahedral ring in a manner which is similar to the fourfold stabilizing assembly. The two tripods are rotated one-sixth of a turn so that every other panel of the dodecahedral ring is stabilized.

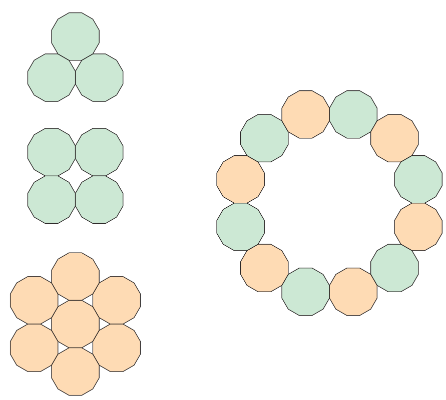




Tripod stablized dodecahedral ring.

Dodecahedral ring assemblies

The dodecahedral rings of identical assemblies can form planar twelvefold assemblies. Each dodecahedral unit will lie one of the vertexes of a regular dodecagon. This is seen at the bottom of the next figure where alternate assemblies have the same color.

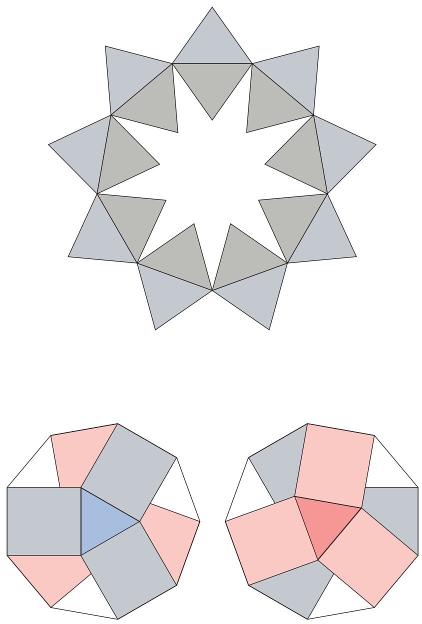


Dodecahedral ring assemblies.
In each of the assemblies the dodecahedral rings lie on the same plane. The joins are square face to square face.

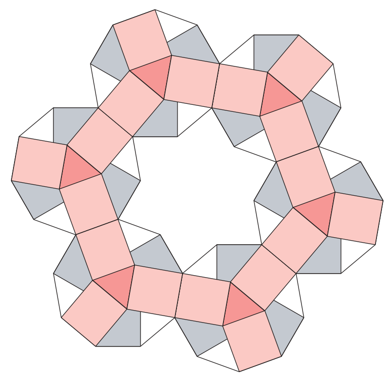
Rings could be joined parallel to the axis of symmetry to form a cylinder.

The figure shows other facially joined dodecahedral rings of three, four, and seven units. The units are at the vertexes of an equilateral triangle, the vertexes of a square, and the vertexes of a regular hexagon. The drawings suggest that the dodecahedral ring assembly is capable of complex structures.

Eighteenfold forms

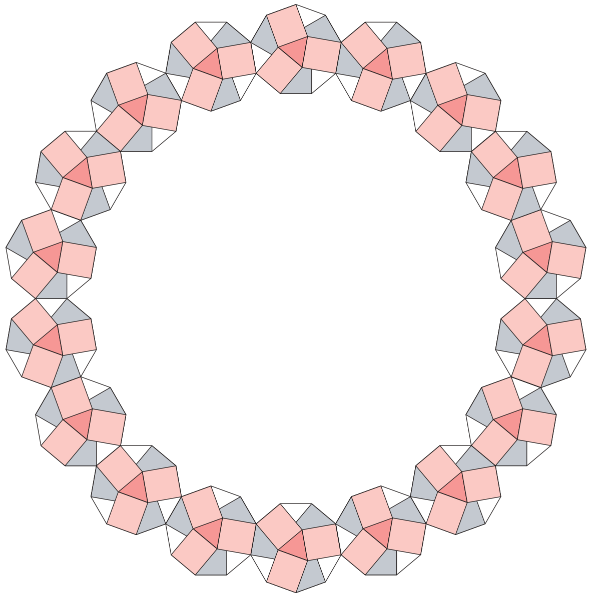


Nonahedral ring assembly
Regular octahedra can form a structurally stable unit consisting of a ring of nine edgially joined regular octahedra and a pair of stabilizing tripodal octahedral assemblies. The equators of the stablized ring project as a regular nonagon, and the square equators are the faces of a regular nonahedral ring.

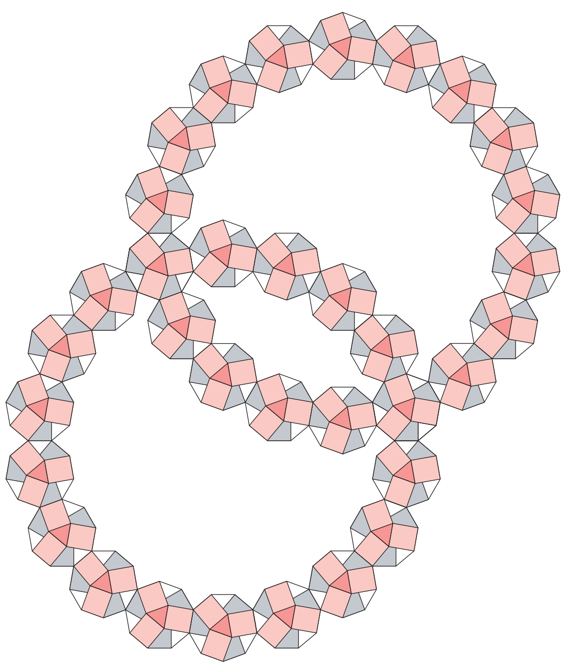


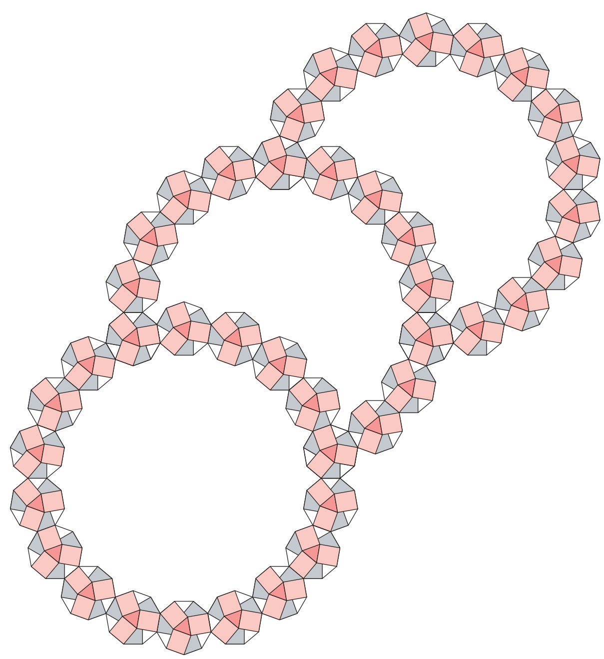
Nonahedral ring assemblies
The nonahedral rings can assemble into a hexagonal ring where each nonahedral unit is at the vertex of a regular hexagon. The units are joined facially.

Tubulin



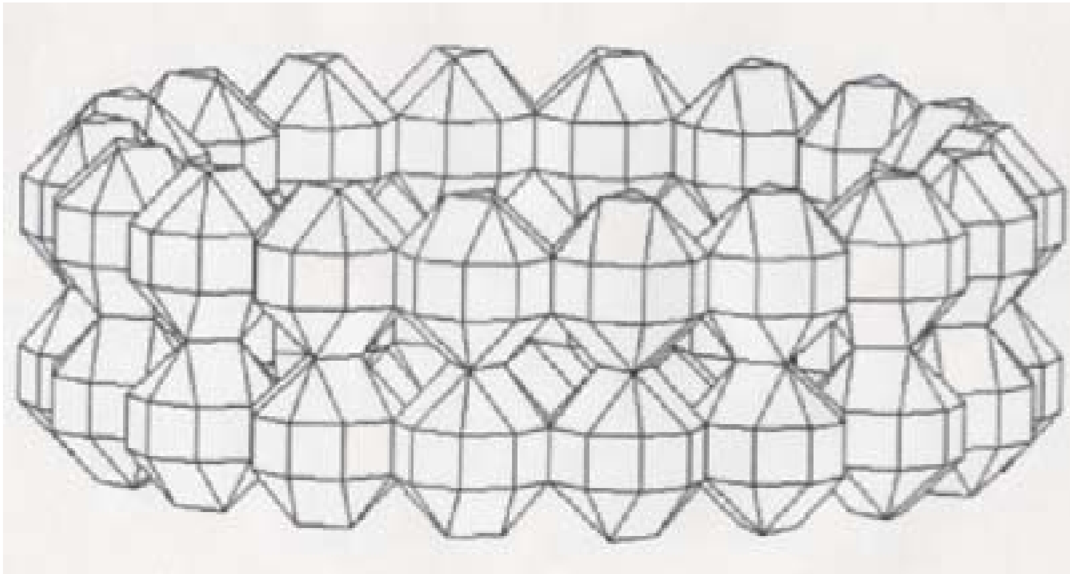
Tubulin
Another ring which they can form is composed of eighteen nonagonal units. This ring has the same symmetry as a ring shown in an article about the centromere and the nonagonal units permit the formation of interpenetrant rings. Two full rings are shown in the next figure which hold two units in common. In the next figure, one full ring has portions of another ring attached to it. This added portion has a portion of a ring attached to it.

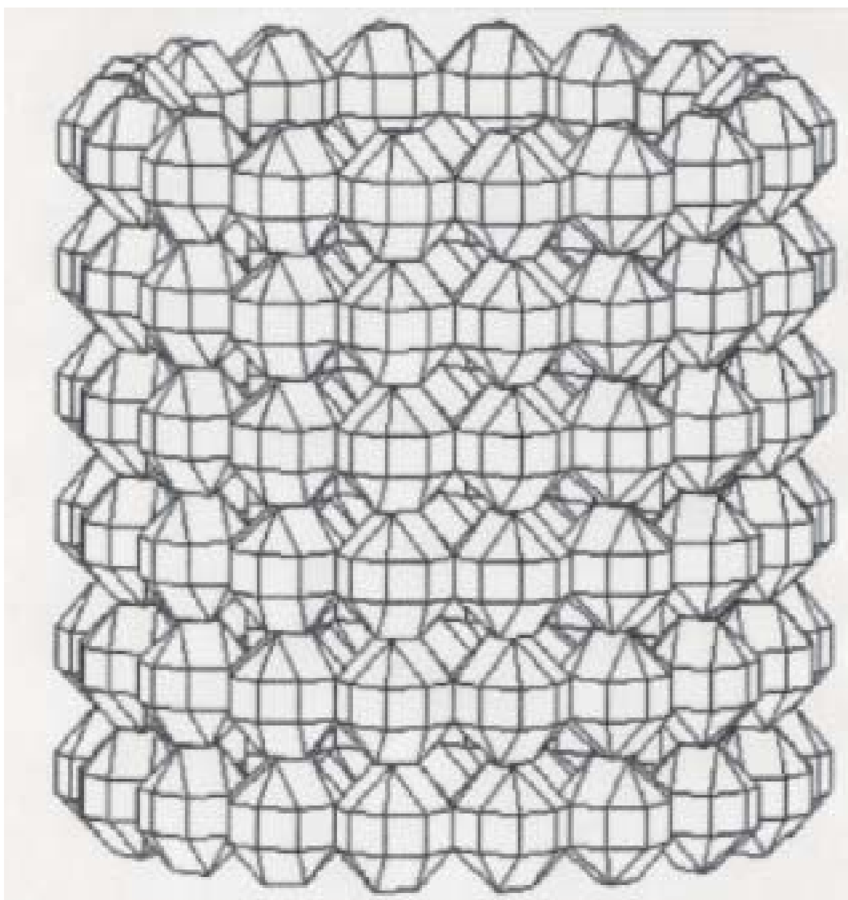




Tubes of nonagonal rings.

The rings of eighteen nonagonal units can be structurally connected parallel to their axis of symmetry to form tubes. The triangular faces of the units are the join surfaces. In tubulin, the individual units will be complex associations of proteins. But, the underlying order will be as octahedral regions in the same orientation as the simple units depicted here. In the figure with two rings stacked one above the other, the view is nearly normal to the stack axis. In the next figure there are six rings in the stack. The view is, again, nearly perpendicular to the stack axis.

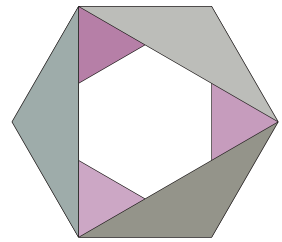




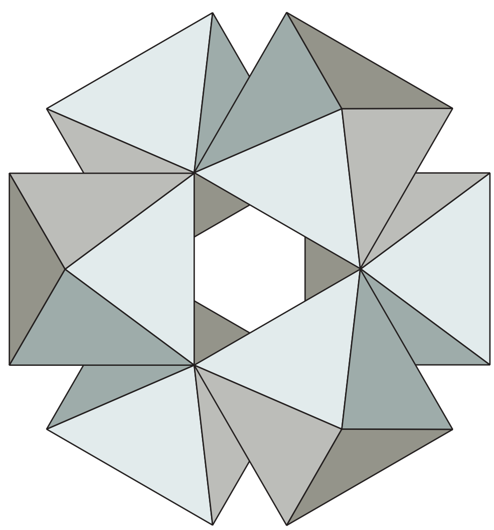
Quasicrystalline rings

Double threefold ring

A structurally stable double threefold ring can be created using six regular octahedral panels which supply the equatorial faces of a regular octahedron. The remaining two faces of the octahedron could be closed by additional octahedral panels. The octahedra of each panel have the same orientation. Panels which are diametrically opposite have octahedra in the same orientation. There are three sets of diametrically opposite panels for the open ended assembly, and there are four sets of diametrically opposite panels for the eight panel assembly.

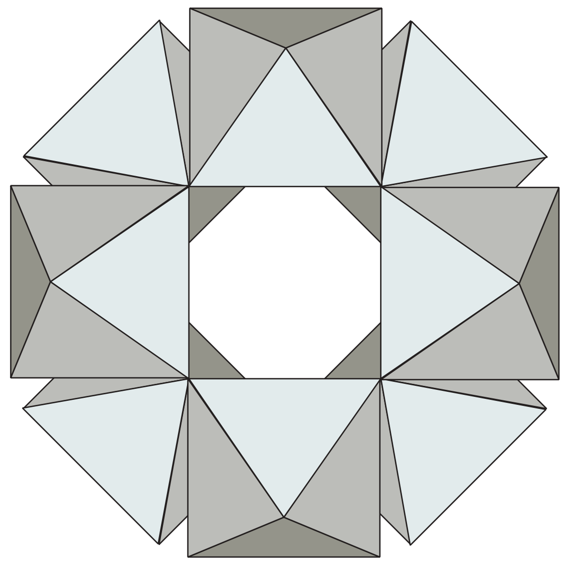


Double threefold ring.
The six equilateral triangular faces are arranged as the equatorial faces of the regular octahedron.

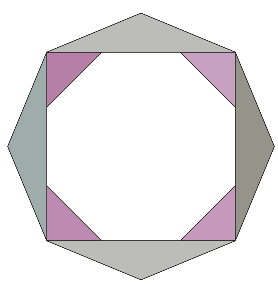


Double threefold ring of octahedra.
Each of the six octahedra acts as facial panel to produce a stable structure.

Double fourfold ring



Double fourfold ring of octahedra.
Each of the eight octahedra acts as a facial panel to produce a stable structure.



Double fourfold ring.
The eight faces are equilateral triangles.

A structurally stable double fourfold ring can be formed of eight regular octahedral panels. Each of the faces is an equilateral triangle. The octahedra of each panel are in the same orientation. The orientation of the octahedra in each panel is unique to that panel. No two panels have octahedra in the same orientation.

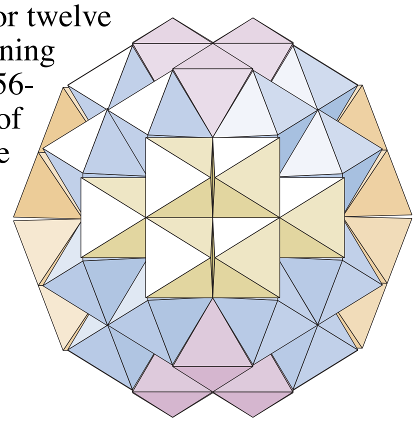
The open faces of the assembly are squares. These could be closed with panels composed of identical octahedra, but the inner vertex of the panel assembly would have to be truncated so that the opposed panels do not obstruct one another.

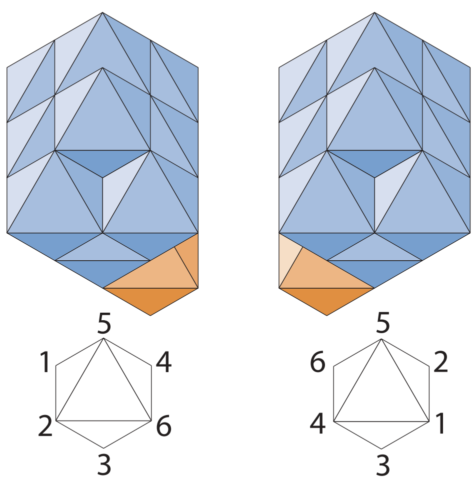
Assemblies of double fourfold rings.

The octahedral panels of a double fourfold ring which is rotated 45° about the axis of symmetry will be oriented so that it can be joined to an unrotated ring. Each of the octahedral panels of the original ring has a counterpart in the rotated ring which is in the same orientation and which is capable of forming a join with it. The panels are at opposite ends of the same diameter.

Ti_8C_{12} icosahedral array

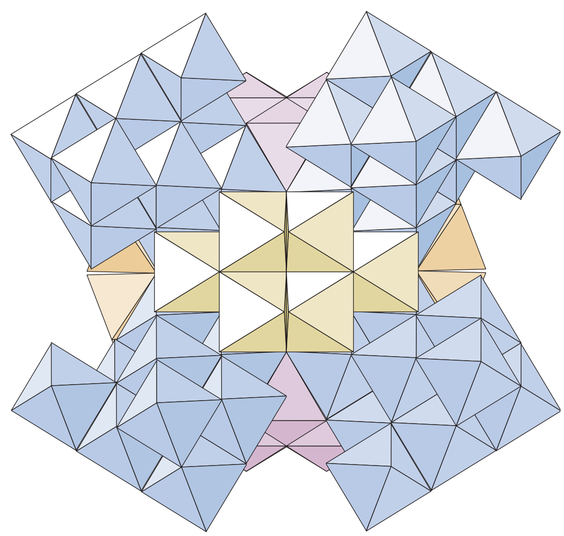
The regular icosahedron has thirty edges each of which lies on a diameter which bisects it and to which it is perpendicular. Three such diameters can be selected which are mutually perpendicular. The three axes have two edges each for six edges. The two faces which define these edges are each contributed by the 123-face of a C-atom. There are six edges times two faces per edge for twelve faces each of which is a C-atom. The remaining eight faces are each supplied by either the 256-face or the 145-face of a Ti-atom. The length of the edge of the icosahedron is twice the edge length of the He-octa. This icosahedral edge length is one half that of the icosahedron which can be built of twenty C_3 units. The latter may be the C_{60}-fullerene.





Ti-atoms with join faces topmost

The Ti-atom has two faces which have the same form as the 623-face of the C-atom—the 526-face and the 541-face. The Ti-atoms depicted above have these faces uppermost. These faces act as panels for the Ti8C12 icosahedron. The Ti-octa of each atom is colored orange here.



Edgial view of Ti_8C_{12} icosahedral array

At the top of the figure is a C_{12} icosahedron viewed edgially. In the figure below it, the blue colored C-atoms have been replaced with blue colored Ti-atoms. An equivalent replacement of four C-atoms on the far side of the assembly produces the Ti_8C_{12} icosahedron.